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A MONOLITHIC APPROACH FOR THE INCOMPRESSIBLE MAGNETOHYDRODYNAMICS EQUATIONS

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Abstract. A numerical algorithm has been developed to solve the incompressible magnetohydrodynamics (MHD) equations in a fully coupled form. The numerical approach is based on the side centered finite volume approximation where the velocity and magnetic filed vector components are defined at the center of edges/faces, meanwhile the pressure term is defined at the element centroid. In order to enforce a divergence free magnetic field, a magnetic pressure is introduced to the induction equation. The resulting large-scale algebraic linear equations are solved using a one-level restricted additive Schwarz preconditioner with a block-incomplete factorization within each partitioned sub-domains. The parallel implementation of the present fully coupled unstructured MHD solver is based on the PETSc library for improving the efficiency of the parallel algorithm. The numerical algorithm is validated for 2D lid-driven cavity flows and backward step problems for both conducting and insulating walls.

1 INTRODUCTION

Magnetohydrodynamics (MHD) deals with the interaction between magnetic fields and the fluid flow. The fluid has to be electrically conducting and non-magnetic in order to interact with the magnetic field. The examples of such fluids are liquid metals, hot or cold plasmas and strong electrolytes. The interaction is a result of Ampere's and Faraday's law and also the Lorentz force is involved in the process. First, the relative motion of the fluid and the magnetic field creates an electromotor force (emf) and electrical currents are induced. Then, these currents induce a secondary magnetic field. Finally, the combined magnetic fields interacts with the induced current density and creates the Lorentz force. Magnetohydrodynamics is important for many applications in engineering and scientific phenomenon such as sunspots, solar flares, interaction between solar winds and Earth's magnetosphere, controlled thermofusion, propulsion, electromagnetic pumps, control of liquid metals, etc. [1, 2].

The mathematical description of incompressible MHD flow includes conservation of mass, conservation of momentum, magnetic induction equation and the divergence-free condition of the magnetic field. The coupled system of MHD equations can be solved using two different coupling strategies. The first one is partitioned (staggered) methods which the equations for fluid and magnetic field are solved separately. The other category is the fully coupled (monolithic) methods. In monolithic approaches the equations for both fields are discretized and solved simultaneously. Staggered approaches provide the freedom to choose optimized solvers for each unknown field. But their convergence rate are slow for fixed point (Picard) iterations and they may diverge for strong interactions (i.e Hartman number greater than unity). The advantage of the monolithic approaches is their robustness but this also leads to computational expense because they require the solution of large systems of coupled non-linear equations. The comparison of both methods can be found in [3].

One of the first numerical studies on magnetohydrodynamics is done by Singh and Lal [4] by solving steady MHD flow in a triangular channel with non-conducting walls by finite difference method for different Hartmann numbers. In 1984, they also employed a finite element formulation with triangular elements for unsteady MHD flow in channels with arbitrary wall conductivity and different Hartmann numbers [5]. Gerbeau [6] employed a stabilized finite element method to solve incompressible MHD equations in two dimensions by using Streamline Upwind Petrov Galerkin (SUPG) method for stabilization. Ni and Li [7] developed a consistent conservative scheme for calculation of the current density and the Lorentz force. They apply the consistent projection method to get the velocity and pressure at time level $n+1$ from the known parameters at time level n on a rectangular staggered mesh and for collocated mesh. Badia et al. [8] proposed a finite element formulation with segregating the velocity and magnetic field for incompressible MHD flow. Shadid et al. [9] proposed a scalable implicit and fully coupled solver for incompressible resistive MHD with stabilized unstructured finite element and Newton-Krylov-AMG. Lin et al. [10] investigated the performance of a fully coupled algebraic multilevel preconditioner for Newton-Krylov solution methods and the performance of the preconditioner is demonstrated for several multiphysics problems including MHD. Cyr et al. [11] proposed and investigated the performance of several candidate block preconditioners for MHD system. Using previously developed preconditioners for Navier-Stokes, and an initial Schur-complement approximation for the magnetic and velocity fields, they showed that the splitting the preconditioner is scalable and competitive with other preconditioners, including a fully coupled algebraic multigrid method. Phillips et al. [12] developed a block conditioner for the finite element discretization of exact penalty formulation of steady fully-coupled MHD in two-dimensions. They employed two types of block preconditioners, one is based on the approximation of Schur compliment and the other one is based on the Newton's method. Adler et al. [13] employed a mixed finite-element discretization of a viscoresistive MHD model with a geometric multigrid preconditioner (monolithic approach).

One of the problems in the solution of MHD equations is to satisfy the divergence-free condition of magnetic field. Severeal methods have been proposed to overcome this problem; adding a diffusion term in the magnetic induction equation [14], constraint transport method [15], using magnetic vector potential formulation, projection method that solves a Poisson equation, staggered grid technique, parabolic divergence cleaning method [16].

In the present work, a numerical method based on the side centered finite volume approximation, where the velocity and magnetic filed vector components are defined at the center of edges/faces, meanwhile the pressure term is defined at the element centroid, is developed to solve incompressible MHD equations in a fully-coupled form. In order to solve the overdetermined system, a magnetic pressure is defined at the cell center and the gradient of this pressure is added to the magnetic induction equation with certain boundary conditions as described in [17], that will lead that pressure to be zero all over the domain while satisfying divergence-free condition of magnetic field. The resulting algebraic system is solved using a one-level restricted additive Schwarz preconditioner with a block-incomplete factorization within each partitioned sub-domains.

2 MATHEMATICAL AND NUMERICAL FORMULATION

The conservation forms of the governing equations for incompressible magnetohydrodynamics (MHD) flow are as follows:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot\left[\rho\mathbf{u}\otimes\mathbf{u} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_m} + \left(p + \frac{B^2}{2\mu_m}\right)\mathbf{I} - \mathbf{T}\right] = 0 \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \frac{1}{\mu_m\sigma}\nabla^2\mathbf{B} + \nabla\cdot\left[-\mathbf{u}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{u}\right] = 0 \tag{2}$$

These equations can be non-dimensionalized as follows using $\mathbf{u} = \mathbf{u}^*U$, $\mathbf{x} = \mathbf{x}^*L$, $t = t^*L/U$, $p = p^*\rho U^2$ and $\mathbf{B} = \mathbf{B}^*B_0$

$$Re\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot\left[Re\mathbf{u}\otimes\mathbf{u} - SRe\mathbf{B}\otimes\mathbf{B} + \left(p + SRe\frac{B^2}{2}\right)\mathbf{I} - \mathbf{T}\right] = 0 \tag{3}$$

$$Re_m\frac{\partial \mathbf{B}}{\partial t} - \nabla^2\mathbf{B} + Re_m\nabla\cdot\left[-\mathbf{u}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{u}\right] = 0 \tag{4}$$

where Re is Reynolds number, Re_m is magnetic Reynolds number and S is the coupling number

$$Re = \frac{\rho UL}{\mu_f}, \quad Re_m = \mu_m\sigma UL, \quad S = \frac{B^2}{\rho\mu_m U^2}$$

where $\mathbf{u}$ is the velocity vector, $\mathbf{B}$ is the magnetic field, $\mathbf{T}$ is the fluid stress tensor, $\mathbf{I}$ is identity matrix, ρ is the fluid density, p is the pressure, σ is the electrical conductivity, μ_m is the magnetic permeability and μ_f is the dynamic viscosity of the fluid.

In order to satisfy the solenoidal property of magnetic field, the gradient of a Lagrange multiplier q introduced to the magnetic induction equation as proposed in [17]. q leads to zero over all the domain therefore becomes a dummy variable by applying proper boundary conditions. Therefore the integral form of incompressible MHD equations that govern the viscous fluid flow of a control volume Ω with boundary $\partial\Omega$ can be written in Cartesian coordinate system in dimensionless form as follows:

$$-\oint_{\partial\Omega_e} \mathbf{n}\cdot\mathbf{u}dS = 0 \tag{5}$$

Momentum equation:

$$\begin{aligned} Re\int_{\Omega_d}\frac{\partial\mathbf{u}}{\partial t}dV + Re\oint_{\partial\Omega_d}[\mathbf{n}\cdot\mathbf{u}]\mathbf{u}dS + \oint_{\partial\Omega_d}\mathbf{n}PdS \\ -\oint_{\partial\Omega_d}\mathbf{n}\cdot\nabla\mathbf{u}dS - \frac{SRe}{Re_m}\oint_{\partial\Omega_d}[\mathbf{n}\cdot\mathbf{B}]\mathbf{B}dS = 0 \end{aligned} \tag{6}$$

Magnetic induction equation:

$$\begin{aligned} Re_m\int_{\Omega_d}\frac{\partial\mathbf{B}}{\partial t}dV + Re_m\oint_{\partial\Omega_d}[\mathbf{n}\cdot\mathbf{u}]\mathbf{B}dS + \oint_{\partial\Omega_d}\mathbf{n}qdS \\ -Re_m\oint_{\partial\Omega_d}[\mathbf{n}\cdot\mathbf{B}]\mathbf{u}dS - \oint_{\partial\Omega_d}\mathbf{n}\cdot\nabla\mathbf{B}dS = 0 \end{aligned} \tag{7}$$

Gauss' law of magnetism states that $\mathbf{B}$ is solenoidal:

$$\oint_{\partial\Omega_e}\mathbf{n}\cdot\mathbf{B}dS = 0 \tag{8}$$

In these equations V is the control volume, S is the control volume surface, $\mathbf{n}$ is the outward normal vector and q is Lagrange multiplier.

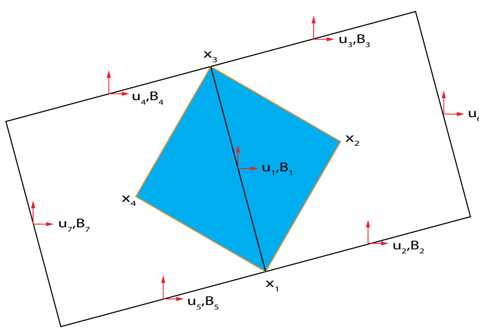


Figure 1: Two-dimensional dual volume

In the present study, semi-staggered Finite Volume Method formulation [18] is applied to the solution of incompressible magnetohydrodynamics equations. The discretization of momentum and magnetic induction equation is done over the dual control volume shown in Figure 1. The local velocity and magnetic field vectors are defined at the center of the each edge. The continuity and divergence of magnetic field equations are integrated over the quadrilateral elements. The pressure and Lagrange multiplier q are defined at the center of elements.The discrezitaziton leads to the following system of algebraic equations

$$\left[\begin{array}{ccc|ccc|c|c} A_{11} & 0 & 0 & A_{14} & 0 & 0 & A_{17} & 0 \\ 0 & A_{22} & 0 & 0 & A_{25} & 0 & A_{27} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{36} & A_{37} & 0 \\ \hline A_{41} & 0 & 0 & A_{44} & 0 & 0 & 0 & A_{48} \\ 0 & A_{52} & 0 & 0 & A_{55} & 0 & 0 & A_{58} \\ 0 & 0 & A_{63} & 0 & 0 & A_{66} & 0 & A_{68} \\ \hline A_{71} & A_{72} & A_{73} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & A_{85} & A_{86} & A_{87} & 0 & 0 \end{array}\right] \left[\begin{array}{c} u \\ v \\ w \\ \hline B_x \\ B_y \\ B_z \\ \hline p \\ \hline q \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ \hline b_4 \\ b_5 \\ b_6 \\ \hline 0 \\ \hline 0 \end{array}\right] \tag{9}$$

where, A_{11},A_{22},A_{33},A_{44},A_{55},A_{66} are the convection diffusion operators,$(A_{17}, A_{27}, A_{37}, A_{48}, A_{58}, A_{68})^T$ are the gradient operator and A_{71},A_{72},$A_{73}A_{85}$,A_{86},A_{87}, are the divergence operator. This fully-coupled system will be solved by a monolithic approach. In order to remove the zero block in the original system, an upper triangular right preconditioner is used

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} & 0 & A_{24} \\ A_{31} & 0 & 0 & 0 \\ 0 & A_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & A_{13} & 0 \\ 0 & I & 0 & A_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \\ p \\ q \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

Then the velocity and magnetic field can be calculates as follows

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{B} \\ p \\ q \end{bmatrix} = \begin{bmatrix} I & 0 & A_{13} & 0 \\ 0 & I & 0 & A_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \\ p \\ q \end{bmatrix} \tag{11}$$

In this work, one-level restricted additive Schwarz preconditioner with a block-incomplete factorization is used within each partitioned sub-domains and the implementation is done by PETSc software package developed at Argonne National Laboratories [19]. For domain decomposition METIS library is employed [20].

3 RESULTS

The developed numerical algorithm is applied to lid-driven cavity and backward facing step problems under the effect of external magnetic field and very good agreements are observed with the results available in the literature.

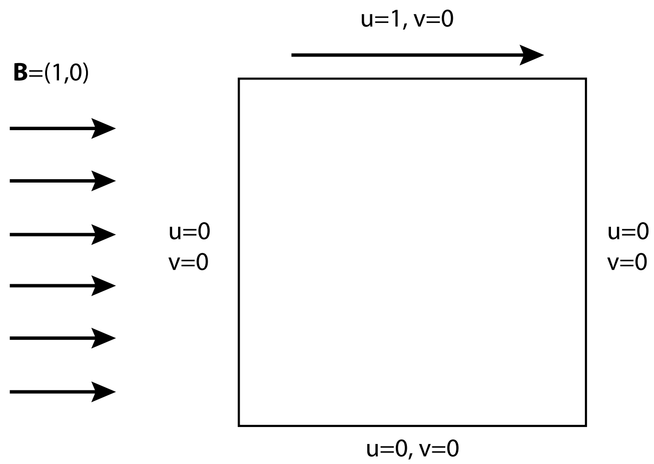


Figure 2: Boundary conditions for cavity problem.

3.1 Lid-Driven Cavity

For this problem, external magnetic field is imposed to classical lid-driven cavity problem. Top wall is moving with velocity $\mathbf{u} = (1,0)$. The walls of the cavity are assumed to be insulating and the magnetic field is in the x-direction, $\mathbf{B} = (1,0)$, see Figure 2.

Different Reynolds numbers and coupling parameters are used for the simulations and compared with the work of Shatrov et al. [21] and Marioni et al. [22]. The streamlines for $Re = 5000$ and $Re_m = 1$ for different coupling parameters are shown in the Figures 3-5. As the coupling number increases, the number of eddies are increased in the cavity. The velocity in the x-direction at the mid-line is shown in the Figure 6 for the upper half o the cavity for $S = 5$ and $S = 30$. Then Reynolds number is increased to 10000 and solved for $S = 0.5$. The velocity streamlines are compared with the reference work [22] in the Figure 7.

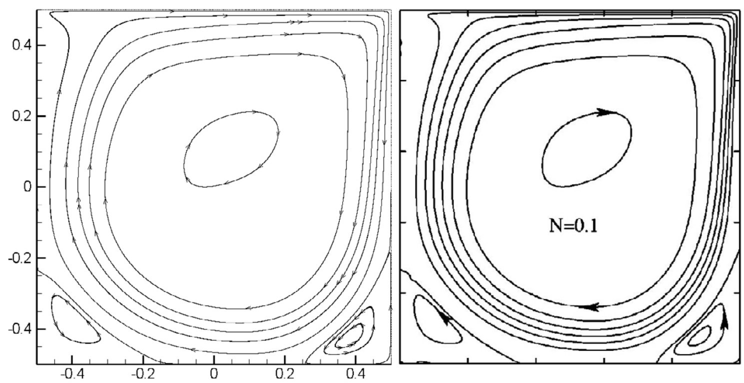


Figure 3: Streamlines at $Re = 5000$, $Re_m = 1$ and $S = 0.1$ for present work (left) and Shatrov et al. (right).

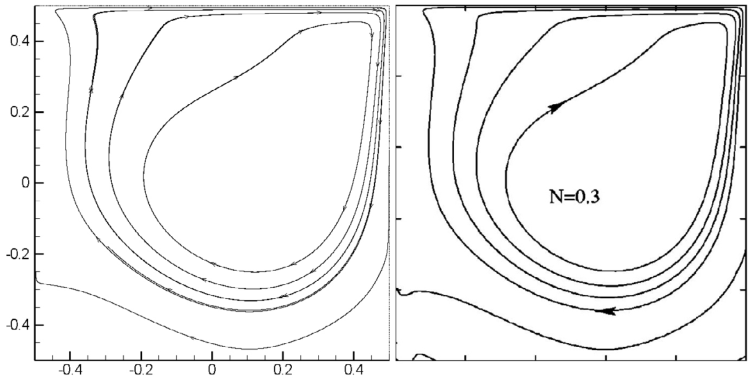


Figure 4: Streamlines at $Re = 5000$, $Re_m = 1$ and $S = 0.3$ for present work (left) and Shatrov et al. (right).

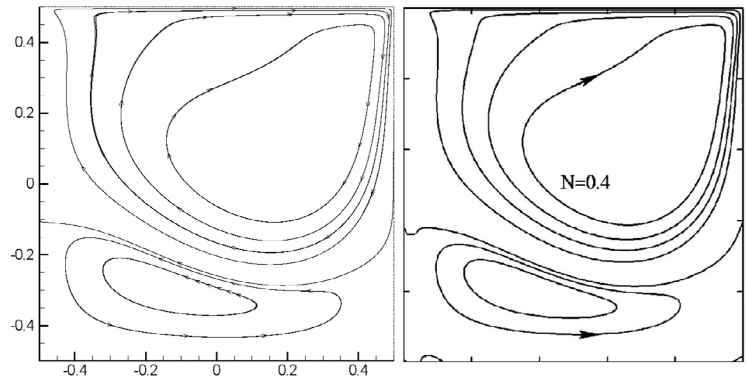


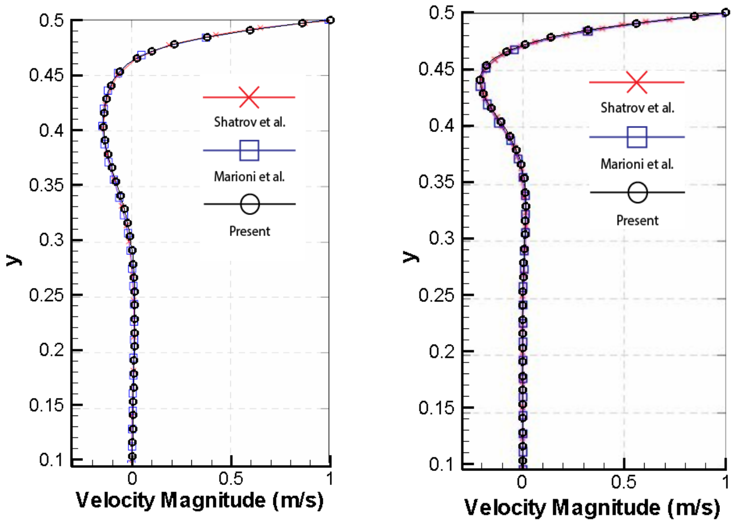
Figure 5: Streamlines at $Re = 5000$, $Re_m = 1$ and $S = 0.4$ for present work (left) and Shatrov et al. (right).

3.2 Backward Facing Step

The first is study on that specific problem is done by Gerbeau [6]. In this problem the walls of the channel are considered to be conducting and transverse magnetic field is applied $\mathbf{B}_0 = (0, 1)$. The same geometry and boundary conditions are used for both velocity and magnetic field as described in [23]. The Reynolds number is $Re = 100$ and magnetic Reynolds number is $Re_m = 10^{-5}$. Different coupling parameters are used to observe the effect of magnetic field on the fluid. As it can be seen in Figure 8 and 9, as the coupling number increases the recirculation after the step decreases.

4 CONCLUSIONS

In this study, a semi-staggered unstructured finite volume method is developed for the solution of incompressible magnetohydrodynamics equations. The components of velocity and



(a) (b)

Figure 6: Velocity profile along mid-line for $Re = 5000$, $Re_m = 1$ (a) $S = 5$, (b) $S = 30$

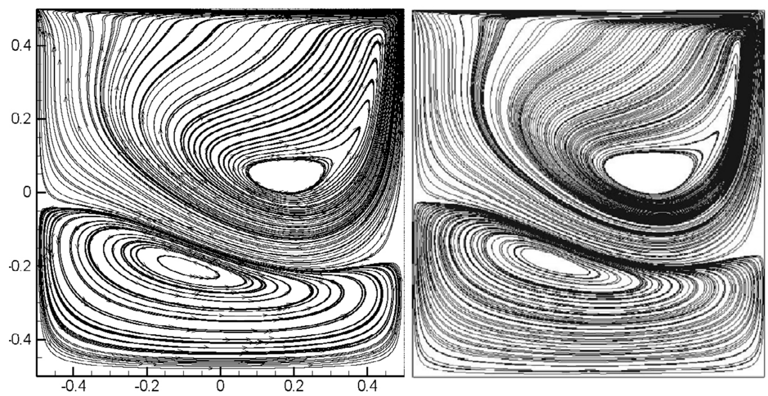


Figure 7: Streamlines at $Re = 10000$, $Re_m = 1$ and $S = 0.5$ for present work (left) and Marioni et al. (right).

magnetic field are defined at the edge centers and pressures are defined at the center of each cell. The resulting system is solved in a fully-coupled approach. The divergence-free condition of the magnetic field is satisfied by introducing the gradient of a scalar multiplier into the magnetic induction equation. One-level restricted additive Schwarz method is used for preconditioning and the implementation of the method is done by using PETSc library. The computational domain is partitioned by METIS libraries. The solver is applied to two-dimensional the lid-driven cavity and backward facing step problems under imposed magnetic field for different Reynolds numbers and coupling parameters.

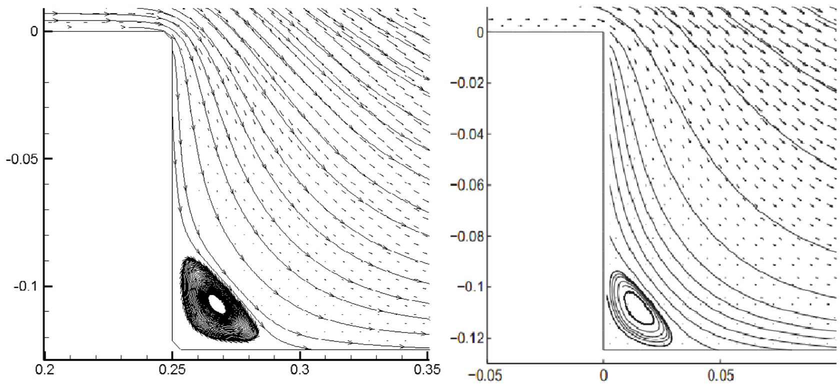


Figure 8: Streamlines for $Re = 100$, $Re_m = 10^{-5}$ and $S = 2.5 \times 10^4$ for present work (left) and Greif et al. (right).

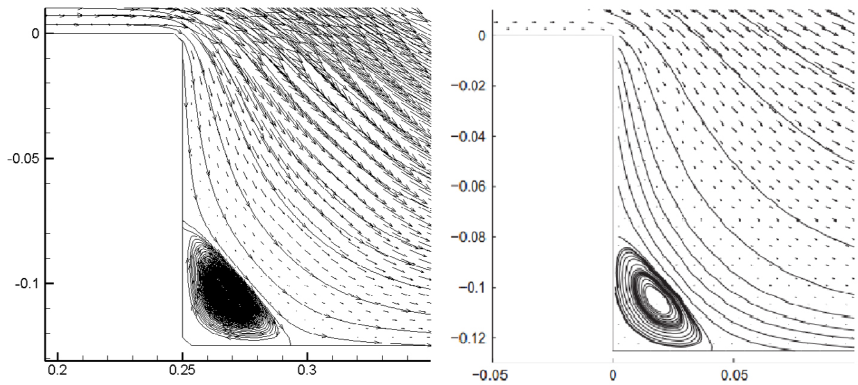


Figure 9: Streamlines for $Re = 100$, $Re_m = 10^{-5}$ and $S = 10^5$ for present work (left) and Greif et al. (right).

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